# Gulf Mathematical Olympiad, Medina, 1437

*English version*

1. Consider sequences $a_0, a_1, a_2, \dots$ of non-negative integers defined by selecting any $a_0, a_1, a_2$ (not all 0) and for each $n \geqslant 3$ letting

$$a_n = |a_{n-1} - a_{n-3}|.$$

   (i) In the particular case that $a_0 = 1, a_1 = 3$ and $a_2 = 2$, calculate the beginning of the sequence, listing $a_0, a_1, \dots, a_{19}, a_{20}$.

   (ii) Prove that for each sequence, there is a constant $c$ such that $a_i \leqslant c$ for all $i \geqslant 0$. *Note that the constant $c$ may depend on the numbers $a_0, a_1$ and $a_2$.*

   (iii) Prove that, for each choice of $a_0, a_1$ and $a_2$, the resulting sequence is eventually periodic. *This means that there is an integer $N$ and a positive integer $p$ such that for all $i \geqslant N$ we have $a_{i+p} = a_i$.*

   (iv) Prove that, the minimum length $p$ of the period described in (iii) is the same for all permitted starting values $a_0, a_1, a_2$ of the sequence.

2. Suppose that the real number $x$ has the property that $x + x^{-1} = 3$.

   (i) Calculate $x^2 + x^{-2}$.

   (ii) Prove that, for each positive integer $n$, the quantity $x^n + x^{-n}$ is a positive integer.

   (iii) Prove that the positive integer $x^{3^{1437}} + x^{-3^{1437}}$ is divisible by at least $1439 \times 2^{1437}$ different positive integers.

3. Consider the acute-angled triangle $ABC$. Let $X$ be a point on the side $BC$, and $Y$ be a point on the side $CA$. The circle $k_1$ on diameter $AX$ cuts $AC$ again at $A'$. The circle $k_2$ on diameter $BY$ cuts $BC$ again at $B'$.

   (i) Let $M$ be the midpoint of $XY$. Prove that $A'M = B'M$.

   (ii) Suppose that $k_1$ and $k_2$ meet at $P$ and $Q$. Prove that the orthocentre of $ABC$ lies on the line $PQ$. *Note that if $ABC$ is a triangle, then the three altitudes of the triangle all pass through a point $H$ which is called the orthocentre of $ABC$.*

4. Suppose that four people $A, B, C$ and $D$ decide to play games of tennis doubles. They might first play the team $A$ and $B$ against the team $C$ and $D$. Next $A$ and $C$ might play $B$ and $D$. Finally $A$ and $D$ might play $B$ and $C$. The advantage of this arrangement is that two conditions are satisfied.

   (a) Each player is on the same team as each other player exactly once.

   (b) Each player is on the opposing team to each other player exactly twice.

   Is it possible to arrange a collection of tennis matches satisfying both condition (a) and condition (b) in the following circumstances?

   (i) There are five players.

   (ii) There are seven players.

   (iii) There are nine players.

*Each problem is worth 10 points.*

*The paper will last 4 hours 30 minutes.*